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EFFECT OF A THIN COATING ON THE PRESSURE DISTRIBUTION IN CONTACT PROBLEMS INVOLVING FRICTIONAL HEAT GENERATION

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A thermoelastic problem for a layer of finite thickness one of whose surfaces is subjected to the action of normal pressure and heat flux is studied. A relationship among vertical displacements of the surface of the layer, the surface temperature, and the disturbing factors is obtained. Corresponding relations are obtained for a layer of small thickness. An axisymmetric contact problem for a rigid heat-conducting base whose surface is coated with a thin elastic layer is studied as an example.

Introduction. Analysis of the stress state of layered media under the joint action of mechanical and temperature loads is very important from the viewpoint of various practical applications, such as protective spraying, layered composite media, thin films, etc. In the present paper, we study the effect of frictional loading on the stress state of a piecewise-homogeneous half-space. The corresponding thermoelastic boundary-value problem is solved using a double integral Fourier transform over spatial variables.

Axisymmetric contact problems for a spherical indenter that interacts with a homogeneous elastic half-space were studied in [1–5]. No solutions for a piecewise-homogeneous half-space have been obtained. We note that isothermal (in the absence of heating) contact problems for a layer and a half-space were studied in [6–8].

1. Solution of a Thermoelastic Boundary-Value Problem for a Piecewise-Homogeneous Half-Space. A piecewise-homogeneous half-space consisting of a layer of thickness h resting on the surface of an elastic half-space is considered. This mechanical system is referred to a rectangular coordinate system x, y, z (Fig. 1). The surface of the layer $z = h$ is exposed to normal pressure p and heat flux q in a finite region Ω. Outside this region, the surface of the layer is not loaded and is heat insulated. The mechanical and thermal contact of the layer and the half-space is ideal.

Investigation of the problem reduces to solution of the Duhamel–Neumann equations [9]

$$(1-2\nu_j)\nabla^2\mathbf{u}^{(j)} + \nabla\operatorname{div}\mathbf{u}^{(j)} = 2\alpha_j(1+\nu_j)a\nabla T^{(j)}, \quad j = 1, 2. \tag{1.1}$$

Here $\mathbf{u}^{(j)} = (au_\xi^{(j)}, au_\eta^{(j)}, au_\zeta^{(j)})$ is the vector of elastic displacement, $T^{(j)}$ is the temperature, $\nabla \equiv (\partial/\partial\xi, \partial/\partial\eta, \partial/\partial\zeta)$, ν_j and α_j are the Poisson ratio and the linear thermal expansion coefficient, $\xi = x/a$, $\eta = y/a$, $\zeta = z/a$, and a is the characteristic linear dimension of the region Ω. Here and below, the subscripts $j = 1$ and 2 refer to the layer and the half-space, respectively.

The temperatures $T^{(j)}$ are obtained by solution of the heat-conduction equations

$$\nabla^2 T^{(j)} = 0, \qquad j = 1, 2. \tag{1.2}$$

The boundary conditions on the surface of the layer $\zeta = h^* = h/a$ are

$$\sigma_{\zeta\zeta}^{(1)} = \begin{cases} -p(\xi,\eta), & (\xi,\eta) \in \Omega, \\ 0, & (\xi,\eta) \notin \Omega; \end{cases} \tag{1.3}$$

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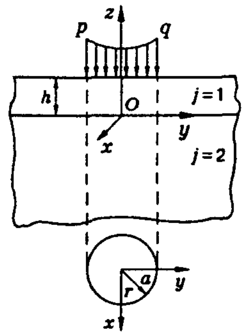

Fig. 1

$$\sigma_{\xi\zeta}^{(1)} = \sigma_{\eta\zeta}^{(1)} = 0; \tag{1.4}$$

$$K_1 \frac{\partial T^{(1)}}{\partial \zeta} = \begin{cases} aq(\xi, \eta), & (\xi, \eta) \in \Omega, \\ 0, & (\xi, \eta) \notin \Omega, \end{cases} \tag{1.5}$$

where K_j ($j = 1$ and 2) are the thermal conductivities. On the interface between the materials $\zeta = 0$ the displacements, stresses, temperatures, and heat fluxes are continuous:

$$u_\xi^{(1)} = u_\xi^{(2)}, \quad u_\eta^{(1)} = u_\eta^{(2)}, \quad u_\zeta^{(1)} = u_\zeta^{(2)}; \tag{1.6}$$

$$\sigma_{\zeta\zeta}^{(1)} = \sigma_{\zeta\zeta}^{(2)}, \quad \sigma_{\xi\zeta}^{(1)} = \sigma_{\xi\zeta}^{(2)}, \quad \sigma_{\eta\zeta}^{(1)} = \sigma_{\eta\zeta}^{(2)}; \tag{1.7}$$

$$T^{(1)} = T^{(2)}; \tag{1.8}$$

$$K_1 \frac{\partial T^{(1)}}{\partial \zeta} = K_2 \frac{\partial T^{(2)}}{\partial \zeta}. \tag{1.9}$$

A particular solution of the thermoelastic equations (1.1) is of the form [10]

$$u_\xi^{(j)} = \frac{\partial \varphi_j}{\partial \xi}, \quad u_\eta^{(j)} = \frac{\partial \varphi_j}{\partial \eta}, \quad u_\zeta^{(j)} = -\frac{\partial \varphi_j}{\partial \zeta}, \tag{1.10}$$

where the thermoelastic potentials φ_j ($j = 1, 2$) are related to the temperatures $T^{(j)}$ by

$$\frac{\partial^2 \varphi_j}{\partial \xi^2} + \frac{\partial^2 \varphi_j}{\partial \eta^2} = \alpha_j (1 + \nu_j) T^{(j)}, \quad \frac{\partial^2 \varphi_j}{\partial \zeta^2} = -\alpha_j (1 + \nu_j) T^{(j)}. \tag{1.11}$$

The thermoelastic potentials φ_j ($j = 1$ and 2) generate the stress fields

$$\begin{gathered} \sigma_{\xi\xi}^{(j)} = -2\mu_j \frac{\partial^2 \varphi_j}{\partial \eta^2}, \quad \sigma_{\eta\eta}^{(j)} = -2\mu_j \frac{\partial^2 \varphi_j}{\partial \xi^2}, \quad \sigma_{\xi\eta}^{(j)} = -2\mu_j \frac{\partial^2 \varphi_j}{\partial \xi \partial \eta}, \\ \sigma_{\zeta\zeta}^{(j)} = \sigma_{\xi\zeta}^{(j)} = \sigma_{\eta\zeta}^{(j)} = 0, \quad j = 1, 2, \end{gathered} \tag{1.12}$$

where μ_j are the shear coefficients.

The system of homogenous differential equations (1.1) is equivalent to the equations [11]

$$\nabla^2 \theta^{(j)} = 0; \tag{1.13}$$

$$\nabla^2 \chi^{(j)} = 0; \tag{1.14}$$

$$\nabla^2 u_\zeta^{(j)} = -d_j \frac{\partial \theta^{(j)}}{\partial \zeta}, \quad d_j = (1 - 2\nu_j)^{-1}, \tag{1.15}$$

where

$$\theta^{(j)} = \frac{\partial u_\xi^{(j)}}{\partial \xi} + \frac{\partial u_\eta^{(j)}}{\partial \eta} + \frac{\partial u_\zeta^{(j)}}{\partial \zeta}; \quad \chi^{(j)} = \frac{\partial u_\eta^{(j)}}{\partial \xi} - \frac{\partial u_\xi^{(j)}}{\partial \eta}.$$

Using condition (1.4), from Eq. (1.14) we have $\chi^{(j)} = 0$. The solution of the differential equations (1.13) and (1.15) is obtained taking an integral Fourier transform over the dimensionless variables ξ and η:

$$\hat{\theta}^{(1)}(\alpha,\beta,\zeta) = C_1(\alpha,\beta)\cosh(s\zeta) + C_2(\alpha,\beta)\cosh[s(h^* - \zeta)]; \tag{1.16}$$

$$\hat{u}_\zeta^{(1)}(\alpha,\beta,\zeta) = -(1/2)d_1C_1(\alpha,\beta)\zeta\cosh(s\zeta) + (1/2)d_1C_2(\alpha,\beta)(h^*-\zeta)\cosh[s(h^*-\zeta)] + D_1(\alpha,\beta)\sinh(s\zeta) + D_2(\alpha,\beta)\sinh[s(h^*-\zeta)]; \tag{1.17}$$

$$\hat{\theta}^{(2)}(\alpha,\beta,\zeta) = C_3(\alpha,\beta)\exp(s\zeta); \tag{1.18}$$

$$\hat{u}_\zeta^{(2)}(\alpha,\beta,\zeta) = -(1/2)d_2C_3(\alpha,\beta)\zeta\exp(s\zeta) + D_3(\alpha,\beta)\exp(s\zeta). \tag{1.19}$$

Here

$$\begin{bmatrix} \hat{\theta}^{(j)} \\ \hat{u}_\zeta^{(j)} \end{bmatrix} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \begin{bmatrix} \theta^{(j)} \\ u_\zeta^{(j)} \end{bmatrix} \exp[-i(\alpha\xi + \beta\eta)]\,d\xi\,d\eta, \quad s = \sqrt{\alpha^2+\beta^2}, \quad i = \sqrt{-1}. \tag{1.20}$$

The stress and strain components in the image space of the integral Fourier transform (1.20) satisfy the relations

$$\hat{u}_\xi^{(j)} = \frac{i}{2\alpha}\Big(\frac{\partial \hat{u}_\zeta^{(j)}}{\partial \zeta} - \hat{\theta}^{(j)}\Big), \quad \hat{u}_\eta^{(j)} = \frac{i}{2\beta}\Big(\frac{\partial \hat{u}_\zeta^{(j)}}{\partial \zeta} - \hat{\theta}^{(j)}\Big); \tag{1.21}$$

$$\hat{\sigma}_{\zeta\zeta}^{(j)} = 2\mu_j\Big(2\frac{\partial \hat{u}_\zeta^{(j)}}{\partial \zeta} + (d_j - 1)\hat{\theta}^{(j)}\Big); \tag{1.22}$$

$$\hat{\sigma}_{\xi\zeta}^{(j)} = \frac{i}{\alpha}\mu_j\Big((s^2 + 2\alpha^2)\hat{u}_\zeta^{(j)} - (1+d_j)\frac{\partial \hat{\theta}^{(j)}}{\partial \zeta}\Big); \tag{1.23}$$

$$\hat{\sigma}_{\eta\zeta}^{(j)} = \frac{i}{\beta}\mu_j\Big((s^2 + 2\beta^2)\hat{u}_\zeta^{(j)} - (1+d_j)\frac{\partial \hat{\theta}^{(j)}}{\partial \zeta}\Big). \tag{1.24}$$

The solution of Eqs. (1.1) is a superposition of solutions (1.10), (1.12), (1.16)–(1.19), and (1.21)–(1.24). Substituting this solution into boundary conditions (1.3), (1.4), (1.6), and (1.7) yields a system of six algebraic equations for the desired functions C_k and D_k (k = 1, 2, 3) of the form

$$[A](C_1, C_2, C_3, D_1, D_2, D_3) = (0, \hat{p}/\mu_1, F_1, F_2, 0, 0). \tag{1.25}$$

The expressions for the coefficients of the matrix A and the right sides of the system are not given because they are cumbersome.

Solving system (1.25), from relations (1.10), (1.11), and (1.17) we find

$$\hat{u}_\zeta^{(1)}(\alpha,\beta,h^*) = \frac{1+d_1}{2s}C_1(\alpha,\beta) + \frac{\alpha_1(1+\nu_1)}{s^2}\frac{\partial \hat{T}^{(1)}}{\partial \zeta}\Big|_{\zeta=h^*}. \tag{1.26}$$

Here

$$C_1(\alpha,\beta) = C_1^*(\alpha,\beta)/C(\alpha,\beta);$$

$$C_1^*(\alpha,\beta) = -\frac{\hat{p}(\alpha,\beta)}{\mu_1}[\mu^*(1+d_1)(1+d_2)\cosh(2\sinh^*) + d_1(\mu^*-1)(d_2+2\mu^*+d_2\mu^*)\sinh^* + (2\mu^*+2d_2+2d_1\mu^*+d_1d_2(1+\mu^{*2}))\sinh(\sinh^*)\cosh(\sinh^*)] - 2sF_1(\alpha,\beta)[d_2(1+d_1)\cosh(\sinh^*) + \mu^*d_1(1+d_2)[\sinh(\sinh^*) + \sinh^*\cosh(\sinh^*)] + d_1(\mu^*+d_2)\sinh^*\sinh(\sinh^*)]$$

$$-2F_2(\alpha,\beta)[\mu^* d_1(1+d_2)\sinh^* \sinh(\sinh^*) + (d_1\mu^* - d_2)\sinh(\sinh^*) + d_1(\mu^* + d_2)\sinh^* \cosh(\sinh^*)];$$

$$C(\alpha,\beta) = d_1(\mu^* + 2d_2 + d_1 d_2)\cosh^2(\sinh^*) + d_1\mu^*(1 + 2\mu^* d_1 + \mu^* d_1 d_2)\sinh^2(\sinh^*)$$

$$+2d_1\mu^*(1+d_1)(1+d_2)\sinh(\sinh^*)\cosh(\sinh^*) - d_1^2(\mu^*-1)(d_2 + 2\mu^* + d_2\mu^*)(\sinh^*)^2 + (d_2 - d_1\mu^*);$$

$$F_1(\alpha,\beta) = -\frac{1}{s^2}\sum_{j=1}^{2}(-1)^j \alpha_j(1+\nu_j)\frac{\partial \hat{T}^{(j)}(\alpha,\beta,\zeta)}{\partial\zeta}\Big|_{\zeta=0};$$

$$F_2(\alpha,\beta) = \sum_{j=1}^{2}(-1)^j \alpha_j(1+\nu_j)\hat{T}^{(j)}(\alpha,\beta,\zeta)\Big|_{\zeta=0}; \quad \mu^* = \mu_1/\mu_2.$$

Solving the heat-conduction boundary-value problem (1.2), (1.5), (1.8), and (1.9), we represent the surface temperature in the image space of the integral Fourier transform as

$$\hat{T}^{(1)}(\alpha,\beta,h^*) = \frac{\hat{q}(\alpha,\beta)a}{K_2 s} + \frac{a}{K_2 s}\left(\frac{1}{K^*} - K^*\right)\frac{\hat{q}(\alpha,\beta)\sinh(\sinh^*)}{\cosh(\sinh^*) + K^*\sinh(\sinh^*)} \qquad (K^* = K_1/K_2). \tag{1.27}$$

We note that in the corresponding isothermal problem ($q = 0$), when $\mu^* \to 0$ relation (1.26) leads to the well-known relation for normal surface displacements of an elastic layer that is rigidly attached to a base [12].

For a layer of small thickness ($h^* \ll 1$), asymptotic analysis of relations (1.26) and (1.27) using the method of [13] leads to the formulas

$$u_\zeta^{(1)}(\xi,\eta,h^*) \approx -\frac{1-\nu_2}{4\pi^2\mu_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp[i(\alpha\xi+\beta\eta)]}{\sqrt{\alpha^2+\beta^2}}\,d\alpha\,d\beta\iint_\Omega p(x,y)\exp[-i(\alpha x+\beta y)]\,dx\,dy$$

$$+\frac{\delta_2 a}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp[i(\alpha\xi+\beta\eta)]}{\alpha^2+\beta^2}\,d\alpha\,d\beta\iint_\Omega q(x,y)\exp[-i(\alpha x+\beta y)]\,dx\,dy - \frac{(d_2+\mu^*)(d_2-d_1\mu^*)}{\mu_1(1+d_1)d_2^2}$$

$$\times h^* p(\xi,\eta) + \left[\frac{2d_1\mu^* - d_2 + d_1 d_2}{(1+d_1)d_2}\left(\delta_1 - \frac{\delta_2}{K^*}\right) + \delta_1 - \delta_2\right]K_1 T^{(1)}(\xi,\eta,h^*); \tag{1.28}$$

$$T^{(1)}(\xi,\eta,h^*) = \frac{a}{4\pi^2 K_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp[i(\alpha\xi+\beta\eta)]}{\sqrt{\alpha^2+\beta^2}}\,d\alpha\,d\beta\iint_\Omega q(x,y)\exp[-i(\alpha x+\beta y)]\,dx\,dy$$

$$+\frac{a}{K_2}\left(\frac{1}{K^*} - K^*\right)h^* q(\xi,\eta) \qquad [\delta_i = \alpha_i(1+\nu_i)/K_i]. \tag{1.29}$$

The last two terms on the right side of relation (1.28) and the last term in (1.29) define the effect of the thin elastic heat-conducting coating. Letting $h^* \to 0$, from (1.28) and (1.29) we obtain a solution for the half-space.

The natural question arises: What is the relative thickness of the layer h^* for which relations (1.28) and (1.29) can be used? It is difficult to give a mathematically rigorously proved answer to this question. To obtain approximate estimates, we consider the auxiliary axisymmetric heat-conduction boundary-value problem (1.2), (1.5), (1.8), and (1.9) for $q(\xi,\eta) = q = \text{const}$, where Ω is a circle of radius a. The exact solution of this problem for $\zeta = h^*$ is of the form

$$T^{(1)}(\rho,h^*) = \frac{aq}{K_2}\int_0^\infty s^{-1}J_1(s)J_0(s\rho)\left[1 + \left(\frac{1}{K^*} - K^*\right)\frac{\sinh(\sinh^*)}{\cosh(\sinh^*) + K^*\sinh(\sinh^*)}\right]ds, \tag{1.30}$$

and the approximate solution obtained by the method of [13] is given by the formula

$$T^{(1)}(\rho,h^*) = \frac{aq}{K_2}\left[\int_0^\infty s^{-1}J_1(s)J_0(s\rho)\,ds + \left(\frac{1}{K^*} - K^*\right)h^*\right], \tag{1.31}$$

where $\rho = \sqrt{\xi^2 + \eta^2}$; $J_0(\cdot)$ and $J_1(\cdot)$ are Bessel functions of the first kind.

Solutions (1.30) and (1.31) coincide provided that

$$\frac{1}{h}\int_0^\infty s^{-1} J_1(s) J_0(s\rho) \frac{\sinh(\sinh^*)}{\cosh(\sinh^*) + K^*\sinh(\sinh^*)}\, ds \approx 1, \quad \rho < 1. \tag{1.32}$$

Thus, relation (1.32) is a criterion for determining values of h^* for which it is possible to use the approximate solution (1.31). Numerical analysis shows that the allowable range of h^* depends greatly on the parameter K^*. Thus, $h^* < 0.2$ for $K^* = 0.1$, $h^* < 0.05$ for $K^* = 1$, and $h^* < 0.01$ for $K^* = 5$. This is explained by the fact that in the Maclaurin series expansion of $\hat{T}^{(1)}(s, h^*)$ in powers of $\sinh^*$, the coefficient of $(\sinh^*)^2$ increases with increase in K^*.

The analogy between formulas (1.28) and (1.29) leads to the conclusion that the effects of the dimensionless parameters K^* and μ^* on the accuracy of a solution are similar. Hence, the proposed method is most effective for a mechanical system in which the rigidity and thermal conductivity of the thin coating are considerably smaller than the corresponding characteristics of the base.

2. An Axisymmetric Contact Problem Involving Frictional Heat Generation. As an example, we consider the interaction of two bodies making contact over a circular area of radius a under the action of a pressing force P. One body moves along the surface of the other at a constant velocity v. The moving body is inhomogeneous: it consists of a rigid heat-conducting base whose surface is coated with a thin elastic layer of thickness h.

Friction leads to heat generation in the contact area by the law

$$q(\rho) = \begin{cases} fvp(\rho), & \rho \leqslant 1, \\ 0, & \rho > 1, \end{cases} \tag{2.1}$$

where f is the friction coefficient, p is the contact pressure, and $\rho = r/a$.

We assume that:

(a) the tangential stresses acting on the contact area do not affect vertical displacements [2, 14];

(b) the immovable body is a rigid heat insulator, and, hence, the heat evolved in the contact area goes to heating of the moving body.

The unknown pressure distribution $p(\rho)$ is found from the contact condition

$$u_\zeta^{(1)} = \frac{\rho^2}{2R} - \Delta, \qquad \rho \leqslant 1 \tag{2.2}$$

($R \gg a$ is the radius of curvature of the moving body and Δ is the approach of the bodies) and the equilibrium condition

$$2\pi a^2 \int_0^1 \rho p(\rho)\, d\rho = P. \tag{2.3}$$

Passing to the limit as $\mu^* \to 0$ and $\alpha_2 \to 0$, from relation (1.28) we obtain the elastic displacement of the surface of the moving body:

$$u_\zeta^{(1)}(\rho, h^*) = -\frac{h^*}{\mu_1(1+d_1)} p(\rho) + \frac{2d_1\alpha_1(1+\nu_1)h^*}{1+d_1} T^{(1)}(\rho, h^*), \quad \rho \geqslant 0. \tag{2.4}$$

Using Eq. (1.29), for the contact temperature we obtain

$$T^{(1)}(\rho, h^*) = \frac{a}{K_2}\left(\frac{1}{K^*} - K^*\right) h^* q(\rho) + \frac{a}{K_2}\int_0^\infty J_0(s\rho)\, ds \int_0^1 \tau q(\tau) J_0(\tau s)\, d\tau, \quad \rho \geqslant 0. \tag{2.5}$$

Substituting relations (2.1) and (2.5) into formula (2.4) and the result obtained into the boundary condition (2.2), we arrive (discarding terms of the order of h^{*2}) at a Fredholm integral equation of the second kind for the contact pressure $p(\rho)$:

$$p(\rho) = \beta^* a \int\limits_0^\infty J_0(s\rho)\,ds \int\limits_0^1 \tau p(\tau) J_0(\tau s)\,d\tau + \frac{\mu_1(1+d_1)}{h^*}\left(\Delta - \frac{a\rho^2}{2R}\right), \quad 0 \leqslant \rho \leqslant 1. \tag{2.6}$$

Here $\beta^* = \delta_1 f v K^*/\gamma$ and $\gamma = (1-2\nu_1)/(2\mu_1)$.

In the case of a moving body with a slightly curved surface ($R \neq 0$), the contact pressure must satisfy the continuity condition $p(1) = 0$, from which we find

$$\frac{\mu_1(1+d_1)}{h^*}\Delta = \frac{\mu_1(1+d_1)a}{2h^*R} - \beta^* a \int\limits_0^\infty J_0(s)\,ds \int\limits_0^1 \tau p(\tau) J_0(\tau s)\,d\tau. \tag{2.7}$$

Substituting (2.7) into (2.6), we write

$$p(\rho) = \frac{\mu_1(1+d_1)a}{2h^*R}(1-\rho^2) + \beta^* a \int\limits_0^\infty [J_0(s\rho) - J_0(s)]\,ds \int\limits_0^1 \tau p(\tau) J_0(\tau s)\,d\tau, \quad 0 \leqslant \rho \leqslant 1. \tag{2.8}$$

Since the integral equation (2.8) is similar in structure to the equation obtained in [5], it is concluded that a critical value of the radius of the area of contact is reached when $P \to \infty$ or $R \to \infty$. Using the procedure proposed in [5], we have $a_{\rm cr} = 2.64/\beta^* = 2.64\gamma/(\delta_1 f v K^*)$.

The solution of Eq. (2.8) is sought in the form

$$p(\rho) = \frac{\mu_1(1+d_1)a}{2h^*R}(1-\rho^2)p^*(\rho),$$

where the function $p^*(\rho)$ satisfies the equation

$$p^*(\rho) = 1 + \beta^* a \int\limits_0^\infty \frac{J_0(s\rho) - J_0(s)}{1-\rho^2}\,ds \int\limits_0^1 \tau p^*(\tau)(1-\tau^2) J_0(\tau s)\,d\tau, \quad 0 \leqslant \rho \leqslant 1. \tag{2.9}$$

To construct a numerical algorithm for solving the integral equation (2.9), we divide the interval $[0,1]$ into N regions by the points $a_m = k/N$ ($m = 0, 1, \ldots, N$), assuming that $p^*(\rho) = p^*_m = \text{const}$ for $a_{m-1} \leqslant \rho \leqslant a_m$. After integration, we arrive at the following system of linear algebraic equations:

$$\sum_{m=1}^{N} b_{km} p^*_m = 1, \qquad k = 1, 2, \ldots, N.$$

Here

$$b_{km} = \delta_{km} - \beta^* a (1-\rho_k^2)^{-1}[B(\rho_k, a_m) - B(\rho_k, a_{m-1})];$$

$$B(\rho, a) = \begin{cases} a(1-a^2)F(1/2,-1/2;1;\rho^2/a^2) + (2/3)a^3F(1/2,-3/2;1;\rho^2/a^2) \\ \quad - (1/2)a^2(1-a^2)F(1/2,1/2;2;a^2) - (1/4)a^4F(1/2,1/2;3;a^2), & \rho < a, \\ (1/2)a^2(1-a^2)\rho^{-1}F(1/2,1/2;2;a^2/\rho^2) + (1/4)a^4\rho^{-1}F(1/2,1/2;3;a^2/\rho^2) \\ \quad - (1/2)a^2(1-a^2)F(1/2,1/2;2;a^2) - (1/4)a^4F(1/2,1/2;3;a^2), & \rho > a; \end{cases}$$

where $\rho = (k-1/2)/N$ ($k = 1, 2, \ldots, N$), δ_{km} is the Kronecker symbol, and F is the Gauss hypergeometric function [15].

Calculations show that the ratio $P_{\rm H}/P$ ($P_{\rm H}$ is the force necessary for formation of a contact area of radius a in the corresponding isothermal Hertz problem) depends linearly on the parameter $\beta^* a$:

$$P_{\rm H}/P = 1 - \beta^* a/2.64.$$

The distribution of the dimensionless contact stresses $p(\rho)/p(0)$ for certain values of the parameter $\beta^* a$ is shown in Fig. 2.

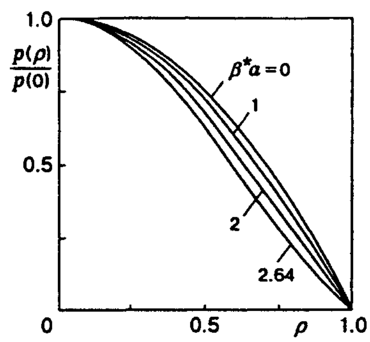

Fig. 2

Furthermore, we study the behavior of the solution of the problem for the case where the moving body has a flat base ($1/R = 0$). Then, the radius a of the contact area is fixed. Representing the solution of Eq. (2.6) in the form

$$p(\rho) = \mu_1(1+d_1)\Delta p^*(\rho)/h^*, \tag{2.10}$$

we obtain

$$p^*(\rho) = 1 + \beta^* a \int_0^\infty J_0(s\rho)\,ds \int_0^1 \tau p^*(\tau) J_0(\tau s)\,d\tau, \quad 0 \leqslant \rho \leqslant 1. \tag{2.11}$$

Substituting relation (2.10) into the statics condition (2.3), we find

$$\Delta = \frac{P_1}{P^*} \quad \left[P_1 = \frac{Ph^*}{2\pi a^2 \mu_1(1+d_1)}, \quad P^* = \int_0^1 \rho p^*(\rho)\,d\rho\right]. \tag{2.12}$$

From formulas (2.12) it follows that the quantity P_1 is fixed and P^* is determined by the solution $p^*(\rho)$ of Eq. (2.11) and, hence, depends on the parameter $\beta^* a$.

The integral equation (2.11) is solved numerically using the foregoing algorithm.

Investigations show that:

(1) for $0 \leqslant \beta^* a < 1.12$ we have $P^* \geqslant 0$, and $P^* \to \infty$ as $\beta^* a \to 1.12$. As a result, $\Delta > 0$, and $\Delta \to 0$ as $\beta^* a \to 1.12$. Thus, for the given range of the parameter $\beta^* a$, force strains dominate over thermal strains;

(2) for $\beta^* a > 1.12$ we have $\Delta < 0$, which indicates predominance of thermal strains over force strains. In addition, if $\beta^* a > 2.64$, separation of the edges of the elastic body from the surface of the half-space occurs. The unknown radius of the contact area is found from the relation $\beta^* a = 2.64$, and the contact-pressure distribution is found from the equation

$$p(\rho) = 2.64 \int_0^\infty [J_0(s\rho) - J_0(s)]\,ds \int_0^1 \tau p(\tau) J_0(\tau s)\,d\tau \quad (0 \leqslant \rho \leqslant 1) \tag{2.13}$$

and the condition of equilibrium of the die (2.3).

The Fredholm equation of the second kind (2.13) has a nontrivial solution, because the value 2.64 coincides with the first characteristic number of this equation.

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